Non-homogeneous equations

In general, suppose

$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$ is linear

If y_g is the general solution of the homogeneous equation, i.e.

$a_2(x)y_g'' + a_1(x)y_g' + a_0(x)y_g = 0$

and y_p is a particular solution of the non-homogeneous equation, i.e.

$a_2(x)y_p'' + a_1(x)y_p' + a_0(x)y_p = f(x)$

then $y = y_g + y_p$ is the general solution of the non-homogeneous equation because

$$
\begin{aligned}
a_2(x)y'' + a_1(x)y' + a_0(x)y &= a_2(x)(y_g'' + y_p'') + a_1(x)(y_g' + y_p') + a_0(x)(y_g + y_p) \\
&= \{a_2(x)y_g'' + a_1(x)y_g' + a_0(x)y_g\} + \{a_2(x)y_p'' + a_1(x)y_p' + a_0(x)y_p\} \\
&= 0 + f(x) \\
&= f(x)
\end{aligned}
$$

So finding the solution of the NH equation requires –

1) Finding y_g, the general solution of the homogeneous equation
2) Find y_p, a particular solution of the NH equation
3) add the 2 above solutions, $y = y_g + y_p$, to find the general solution of the NH equation.

a couple of things to notice –

- We can generally only solve the homogeneous equation with constant coefficients $ay'' + by' + cy = 0$ so we are restricted to NH equations of the form $ay'' + by' + cy = f(x)$.
- This generalizes to equations of any order but since we can, in general, only solve the characteristic equation for 2^{nd} order we restrict ourselves to 2^{nd} order. This is fine for equations of physics and engineering.

The Method of Undetermined Coefficients (Judicious Guessing)

This method works for $f(x)$ of the form of polynomials, sines, cosines, and exponentials. It depends on the fact that if $f(x)$ has one of these forms then all of its derivatives have the same form.

Ex: Find the general solution of $3y''+y'-2y=2\cos x$ where $y=y(x)$.

First find y_g, the solution of the homogeneous equation

$$3y''+y'-2y=2\cos x$$

If $y_g=e^{kx}$ then

$$3k^2+k-2=0$$

$$(3k-2)(k+1)=0$$

$$k=\tfrac{2}{3} \text{ or } k=-1$$

and $y_g=c_1e^{\frac{2}{3}x}+c_2e^{-x}$

Now we need y_p, a particular solution of the NH equation

$$3y''+y'-2y=2\cos x$$

You might guess that y_p is a cosine but, even though the 2nd derivative of the cosine is the cosine, the first derivative will give you a sine which can't be cancelled. So we must include a sine and we guess

$$y_p=A\cos x+B\sin x \text{ where } A, B \text{ are constants. Then}$$

$$y_p'=-A\sin x+B\cos x$$

$$y_p''=-A\cos x-B\sin x$$

If y_p is a solution then

$$3(-A\cos x-B\sin x)+(-A\sin x+B\cos x)-2(A\cos x+B\sin x)=2\cos x$$

or $\quad(-5A+B)\cos x+(-A-5B)\sin x=2\cos x$

Equating coefficients of like terms –

$$-5A+B=2$$
$$-A-5B=0$$

Solving for A & B we find

$$-26A=10$$

$$A=-5/13,\quad B=\tfrac{1}{13}$$

$y_p=\frac{-5}{13}\cos x+\frac{1}{13}\sin x$ and

$y=y_g+y_p=c_1e^{\frac{2}{3}x}+c_2e^{-x}-\frac{5}{13}\cos x+\frac{1}{13}\sin x$ is the general solution of the NH equation!

ex: $f(x)$ = polynomial

$$y'' + y' + y = x^2$$

first solve the homogeneous eqn

$$y'' + y' + y = 0$$

$$k^2 + k + 1 = 0$$

$$k = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$$

so $y_g = e^{-\frac{1}{2}x}\left\{c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$

Now to find y_p we need a function such that the function and its derivatives add to a 2nd degree polynomial. So we guess that y_p is a polynomial of degree 2 –

$y_p = A + Bx + Cx^2$ where A, B, C are unknown

$y_p' = B + 2Cx$

$y_p'' = 2C$

and if y_p satisfies the NH equation –

$$y_p'' + y_p' + y_p = 2C + B + 2Cx + A + Bx + Cx^2 = x^2$$

Collecting like terms we find

$$Cx^2 + (B + 2C)x + (A + B + 2C) = x^2$$

so $C = 1$

$B + 2C = B + 2 = 0$

$B = -2$

$A + B + 2C = A - 2 + 2 = 0$

$A = 0$

and $y_p = x^2 - 2x$ satisfies the NH equation

and $y = y_g + y_p$

$$\boxed{y = e^{-\frac{1}{2}x}\left\{c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right\} + x^2 - 2x}$$

is the general solution of the NH equation.

4: $f(x) =$ exponential

$$y'' - y' - 2y = e^{3x}$$

$$k^2 - k - 2 = 0$$

$$(k-2)(k+1) = 0$$

$$k = 2, -1$$

So $y_g = c_1 e^{2x} + c_2 e^{-x}$ is the solution of the homogeneous equation

For y_p we need a function such that the function and its derivatives are exponential. That would be an exponential.

So we guess

$$y_p = Ae^{3x}, \quad A = \text{unknown}$$

$$y_p' = 3Ae^{3x}$$

$$y_p'' = 9Ae^{3x}$$

and $$9Ae^{3x} - 3Ae^{3x} - 2Ae^{3x} = e^{3x}$$

$$4Ae^{3x} = e^{3x}$$

$$A = \frac{1}{4}$$

So $y_p = \frac{1}{4}e^{3x}$

and $y = y_g + y_p$

$\boxed{y = c_1 e^{2x} + c_2 e^{-x} + \frac{1}{4}e^{3x}}$ is the general solution of the NH equation

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The three cases above are the base cases –

- If $f(x) =$ polynomial of degree n, guess $y_p =$ polynomial of degree n.
- If $f(x) =$ exponential, guess $y_p =$ exponential
- If $f(x) =$ SINE or $f(x) =$ COSINE, guess $y_p =$ sum of SINE and COSINE

But... things do get more complicated!

$\underline{*}$: It may occur that $f(x)$ is not linearly independent of the general solution y_g.

Consider $y''-3y'+2y=e^x$

$$k^2-3k+2=0$$
$$(k-2)(k-1)=0$$
$$k=1\ ;\ k=2$$
$$y_g=C_1e^{2x}+C_2e^x$$

Now notice that if we assume $y_p=Ae^x$

then $y=y_g+y_p=C_1e^{2x}+C_2e^x+Ae^x$
$$=C_1e^{2x}+C_2e^x$$

and we have only y_g because $f(x)=e^x$ is not independent of y_g.

So we take a hint from reduction of order and assume $y_p=Axe^x$.
This is justified by noticing that all derivatives of a polynomial times an exponential have that same form.

So if $y_p=Axe^x$
$$y_p'=Axe^x+Ae^x$$
$$y_p''=Axe^x+Ae^x+Ae^x$$

and
$$y_p''-3y_p'+2y_p=Axe^x+2Ae^x-3Axe^x-3Ae^x+2Axe^x=e^x$$
$$-Ae^x=e^x$$
$$A=-1$$
$$y_p=-xe^x$$

and $y=y_g+y_p$

$\boxed{y=C_1e^{2x}+C_2e^x-xe^x}$ is the general solution.

So, the general rule is –

1) Write the assumed form of y_p based on $f(x)$,
2) If any term of y_p is <u>not</u> independent of y_g then multiply your assumed form of y_p by the smallest power of x which makes that assumed form independent of y_g.

*! $f(x)$ may be a product of polynomials, sines, cosines, exponentials.
That's okay, just assume y_p is of the same form because all derivatives will also have that form.

$$y'' - 3y' - 4y = 3xe^{2x}$$

Find y_g -

$$k^2 - 3k - 4 = 0$$
$$(k-4)(k+1) = 0$$
$$k = 4, -1$$
$$y_g = c_1 e^{4x} + c_2 e^{-x}$$

Now notice that if $f(x) = x$ we would assume $y_p = Ax + B$.
if $f(x) = e^{2x}$ we would assume $y_p = Ae^{2x}$.
Since $f(x) = 3xe^{2x}$ we assume y_p = product of the 2 individual forms -

$$y_p = (Ax+B)e^{2x}$$
$$y_p' = 2(Ax+B)e^{2x} + Ae^{2x}$$
$$y_p'' = 4(Ax+B)e^{2x} + 2Ae^{2x} + 2Ae^{2x} = 4(Ax+B)e^{2x} + 4Ae^{2x}$$

So, substituting into the NH equation -

$$4(Ax+B)e^{2x} + 4Ae^{2x} - 6(Ax+B)e^{2x} - 3Ae^{2x} - 4(Ax+B)e^{2x} = 3xe^{2x}$$
$$(4A - 6A - 4A)xe^{2x} + (4B + 4A - 6B - 3A - 4B)e^{2x} = 3xe^{2x}$$

so $-6A = 3$
$$A = -\frac{1}{2}$$

and $A - 6B = 0$
$$B = \frac{A}{6} = -\frac{1}{12}$$

so $y_p = \left(-\frac{1}{2}x - \frac{1}{12}\right)e^{2x}$
$$= -\frac{1}{2}xe^{2x} - \frac{1}{12}e^{2x}$$

and $\boxed{y = c_1 e^{4x} + c_2 e^{-x} - \frac{1}{2}xe^{2x} - \frac{1}{12}e^{2x}}$

So if $f(x)$ = product, just assume y_p is also a product but be careful, don't include too many unknowns. Above, you may be tempted to assume $y_p = (Ax+B)Ce^{2x}$ but $(Ax+B)Ce^{2x} = (ACx + BC)e^{2x}$
$$= (Ax+B)e^{2x}$$
If you include too many unknown you complicate the problem and may not have enough conditions to find all unknowns.

the simplest complication is when $f(x)$ is a sum of above terms, i.e.

$$ay'' + by' + cy = f(x) + g(x)$$

suppose y_{P_1} solves $ay''_{P_1} + by'_{P_1} + cy_{P_1} = f(x)$ and

y_{P_2} solves $ay''_{P_2} + by'_{P_2} + cy_{P_2} = g(x)$

then $a(y_{P_1} + y_{P_2})'' + b(y_{P_1} + y_{P_2})' + c(y_{P_1} + y_{P_2})$

$$= ay''_{P_1} + by'_{P_1} + cy_{P_1} + ay''_{P_2} + by'_{2} + cy_{P_2}$$

$$= f(x) + g(x)$$

so simply add the 2 assumed forms for each non-homogeneous term in the sum (don't forget to correct for dependence!)

as an example, $y'' + y' - 2y = xe^x + x^2 + 1$

$$k^2 + k - 2 = 0$$
$$(k+2)(k-1) = 0$$
$$k = -2,\ k = 1$$
$$y_g = c_1 e^{-2x} + c_2 e^x$$

for the first NH term, xe^x, we assume $y_P = (Ax+B)e^x$ but the term Be^x is <u>not</u> independent of y_g so we correct by multiplying by the smallest power of x which makes it independent, that is we assume $y_{P_1} = x(Ax+B)e^x = (Ax^2+Bx)e^x$

And for the 2^{nd} term we assume

$y_{P_2} = Cx^2 + Dx + E$, a polynomial of degree 2.

then y_P is just the sum of y_{P_1} and y_{P_2} –

$$y_P = (Ax^2+Bx)e^x + Cx^2 + Dx + E$$

you can solve this if you want, you should find five equations in five unknowns.

A chart of assumed forms of y_P for a given $f(x)$ is attached.

TABLE 3.1 THE PARTICULAR SOLUTION OF $ay'' + by' + cy = g(x)$

$g(x)$	$y_p(x)$
$P_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$	$x^s(A_0x^n + A_1x^{n-1} + \cdots + A_n)$
$P_n(x)e^{\alpha x}$	$x^s(A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{\alpha x}$
$P_n(x)e^{\alpha x}\begin{cases}\sin\beta x\\ \cos\beta x\end{cases}$	$x^s\left[\left(A_0x^n + A_1x^{n-1} + \cdots + A_n\right)e^{\alpha x}\cos\beta x + \left(B_0x^n + B_1x^{n-1} + \cdots + B_n\right)e^{\alpha x}\sin\beta x\right]$

Notes. Here s is the smallest nonnegative integer ($s = 0$, 1, or 2) which will insure that no term in $y_p(x)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation, respectively.